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Supersymmetry of a multi-boson Hamiltonian through the Higgs algebra

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Abstract. We consider a Hamiltonian describing multiphoton processes of scattering and it is then realized in terms of the generators of the Higgs algebra in order to put in evidence the double degeneracy of the corresponding energies. The supersymmetry of such a Hamiltonian is then proved.

1. Introduction

Kinematical and dynamical symmetries—associated with vector fields generating Lie algebras—play a prominent role in physics. In particular, the irreducible representations of such Lie algebras can be used [1] in order to put in evidence some physical spectra (and the corresponding eigenfunctions) of the concerned *linear* models. This method is usually referred to as the group theoretical approach. However, many physical systems cannot be described in such a way because of their *nonlinearity*. In other words, we need an extension of the ordinary Lie algebras in order to handle these systems. This extension does exist and is known as the concept of deformed (nonlinear) algebras. After Karassiov and Klimov [2] in particular, we can characterize these deformations by the following commutation relations

$$[E_a, E_b] = \sum_c C^c_{ab} E_c \tag{1}$$

$$[E_a, V_\alpha] = \sum_\beta \tau^\beta_{a\alpha} V_\beta \tag{2}$$

$$[V_\alpha, V_\beta] = f_{\alpha\beta}(E_c). \tag{3}$$

So, the relation (1) means that the operators E_c generate a usual Lie algebra while the other relations refer to irreducible tensor operators V_α commuting to give back *nonlinear* functions $f_{\alpha\beta}$ of the Lie generators E_c. A particularly interesting example of such a deformed algebra is the so-called Higgs algebra [3], the (cubic) algebra of the symmetries of a two-dimensional harmonic oscillator in a curved space. It is generated by the diagonal operator J_3 and the two scaling operators $J_\pm$ such that the corresponding commutation relations are

$$[J_3, J_\pm] = \pm J_\pm \tag{4}$$

$$[J_+, J_-] = 2J_3 + 8\beta J_3^3 \tag{5}$$

β being a real parameter (related to the curvature of the space in [3]). The irreducible representations of this Higgs algebra have already been analysed [4, 5] leading to many

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new possibilities with respect to the representations of the usual $sl(2, R)$ (corresponding to $\beta = 0$, cf [6]).

The purpose of this paper is to prove that the Higgs algebra can be exploited, in a flat space, in order to point out the spectrum and the eigenfunctions of a *nonlinear* Hamiltonian [7] describing multiphoton processes of scattering, namely

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1^\dagger)^n a_2^m + g^* a_1^n (a_2^\dagger)^m \qquad 0 \leqslant m \leqslant n \tag{6}$$

where $a_j, a_j^\dagger$ $(j = 1, 2)$ are bosonic annihilation and creation operators with frequencies ω_j and g is a coupling constant. We then note a twofold degeneracy of the concerned energies in some cases, a signal for supersymmetry [8] in quantum mechanics and, effectively, we prove that the Hamiltonian (6) is supersymmetric.

The contents are then distributed as follows. In section 2, we relate the Hamiltonian (6) to the Higgs algebra and find the corresponding spectrum. Section 3 is devoted to the supersymmetric features of the Hamiltonian (6). We consider two specific examples in section 4 and conclude with some comments in section 5.

2. The Higgs algebra in quantum optics

Let us rewrite the Hamiltonian (6) as

$$H = (\omega_1 + \omega_2)\left(\sqrt{C + \tfrac{1}{4}} - \tfrac{1}{2}\right) + 2(\omega_1 - \omega_2) J_3 + g J_+ + g^* J_- \tag{7}$$

where the operators J_3 and $J_\pm$ defined by

$$J_+ = (a_1^\dagger)^n a_2^m \qquad J_- = a_1^n (a_2^\dagger)^m \tag{8}$$

$$J_3 = \frac{1}{m+n}(a_1^\dagger a_1 - a_2^\dagger a_2) \tag{9}$$

generate a deformed $sl(2, R)$ algebra

$$[J_3, J_\pm] = \pm J_\pm \tag{10}$$

$$[J_+, J_-] = f(J_3) \tag{11}$$

and C is the Casimir operator of the usual $sl(2, R)$ (in order to avoid confusion, let us say that it is generated by the operators $j_3, j_\pm$)

$$C = j_- j_+ + j_3(j_3 + 1). \tag{12}$$

The only non-trivial context for which the Higgs algebra appears (that is to say, the analytic function f in (11) is the one introduced in (5)) is $n = m = 2$. In this case, we have

$$\beta = -\frac{2}{2j^2 + 2j - 1} \qquad j = 0, \frac{1}{2}, 1, \ldots. \tag{13}$$

These values of β are allowed in the irreducible representations constructed in [5] *only* and they lead to the relations $(m = -j, \ldots, +j)$

$$J_3|j, m\rangle = \frac{m}{2}|j, m\rangle \tag{14}$$

$$J_+|j, m\rangle = \sqrt{(j-m)(j+m+1)(j-m-1)(j+m+2)}\,|j, m+2\rangle \tag{15}$$

$$J_-|j, m\rangle = \sqrt{(j+m)(j+m-1)(j-m+1)(j-m+2)}\,|j, m-2\rangle \tag{16}$$

or, in other words, to

$$J_3 = \tfrac{1}{2} j_3 \qquad J_\pm = (j_\pm)^2. \tag{17}$$

Such relations can be used to determine the energies of the Hamiltonian (7). Indeed, let us introduce the eigenfunctions of (7) as

$$|\psi_k\rangle = \sum_{m=-j}^{j} c_m^{(k)} |j, m\rangle \tag{18}$$

and consider the equation

$$H \mid \psi_k\rangle = E_k \mid \psi_k\rangle. \tag{19}$$

We can, taking account of (7), (14)–(16), (18) and (19), obtain the coefficients $c_m^{(k)}$ as well as the associated eigenvalues E_k through

$$\begin{aligned} E_k c_m^{(k)} = {} & m c_m^{(k)}(\omega_1 - \omega_2) + j c_m^{(k)}(\omega_1 + \omega_2) \\ & + g c_{m-2}^{(k)} \sqrt{(j+m)(j+m-1)(j-m+1)(j-m+2)} \\ & + g c_{m+2}^{(k)} \sqrt{(j-m)(j-m-1)(j+m+1)(j+m+2)}. \end{aligned} \tag{20}$$

Evidently it is possible to solve this system in a general way, but what is interesting for our purpose is to create a maximal symmetry situation. In this case, we know [9] that the angular frequencies have to be equal and we thus take

$$\omega_1 = \omega_2 = \omega. \tag{21}$$

Moreover, we can concentrate on the real g context, the nonlinear part of (7) then describing isotropic paramagnets in two dimensions [10]. It is not a loss of generality as we have

$$\exp\left(\frac{\mathrm{i}\pi}{2} J_3\right)(J_+ + J_-)\exp\left(-\frac{\mathrm{i}\pi}{2} J_3\right) = \mathrm{i}(J_+ - J_-). \tag{22}$$

Finally, we take j to be a half-integer because it is the only case leading to degeneracies (and thus to symmetries) in the spectrum.

We then obtain for each fixed value of j a *twofold* degeneracy of *all* the energies associated with (7), that is

$$E_k = 2\omega j + g\lambda_k \qquad k = 1, 2, \ldots, j + \tfrac{1}{2} \tag{23}$$

where λ_k is anyone of the $(j + \frac{1}{2})$ different solutions of

$$\begin{aligned} [F(A_k, j, \lambda)]^2 \equiv {} & \Bigg[\lambda^{j+\frac{1}{2}} - \sum_{k=1}^{j-\frac{1}{2}} A_k^2 \lambda^{j-\frac{3}{2}} + \Bigg(\sum_{\substack{k<l \\ |k-l|\neq 2}}^{j-\frac{1}{2}} A_k^2 A_l^2 - A_{j-\frac{3}{2}}^2 A_{j-\frac{1}{2}}^2\Bigg)\lambda^{j-\frac{7}{2}} \\ & - \Bigg(\sum_{\substack{k<l<p \\ |k-l|\neq 2 \\ |k-p|\neq 2 \\ |l-p|\neq 2}}^{j-\frac{1}{2}} A_k^2 A_l^2 A_p^2 - \sum_{k=1}^{j-\frac{9}{2}} A_k^2 A_{j-\frac{3}{2}}^2 A_{j-\frac{1}{2}}^2\Bigg)\lambda^{j-\frac{11}{2}} \ldots\Bigg]^2 = 0. \end{aligned} \tag{24}$$

In this last relation, the quantities A_k are defined by

$$A_k = (k(k+1)(2j-k)(2j-k+1))^{1/2} \qquad k = 1, 2, \ldots, j - \tfrac{1}{2}. \tag{25}$$

Evidently it is the square in (24) which is the *ad hoc* (remarkable) signal for the twofold degeneracy.

Because of this twofold degeneracy, we can reasonably hope that the Hamiltonian (7) is supersymmetric. The proof of this assertion is performed in the next section. However, as supersymmetry asks for a semi-definite positive Hamiltonian, we assume from now on that the energies (23) are positive ones (if not, we just have to make a shift).

3. Supersymmetry in quantum optics

In order to prove that the Hamiltonian (7) is supersymmetric, we have to find N self-adjoint operators, namely the supercharges, generating the Lie superalgebra $sqm(2)$ [8]

$$\{Q_k, Q_l\} = 2\delta_{kl} H \qquad k, l = 1, 2, \ldots, N \tag{26}$$

where the curly brackets evidently refer to anticommutators. We have not been able to find these operators Q_k in the Higgs representation without fixing j and that is why we are going to consider another (more easily handled) representation of the Hamiltonian (7). To this aim, we first note that it is possible to construct a transformation S such that

$$H' = S^{-1} H S \tag{27}$$

where

$$H' = \mathrm{diag}\,(E_1, E_2, \ldots, E_{j+\frac{1}{2}}, E_1, E_2, \ldots, E_{j+\frac{1}{2}}) \tag{28}$$

the energies E_k being known through (23). This (non-unitary) transformation S is indeed given by

$$S = \sum_{k,l=1}^{j+\frac{1}{2}} (\psi_{2k-1}^{(l)} e_{2k+1,l} + \psi_{2k}^{(l)} e_{2k,j+l+\frac{1}{2}}) \tag{29}$$

where the notation $e_{k,l}$ stands for a $(2j+1)$ by $(2j+1)$ matrix with zeros everywhere with the exception of a unit at the intersection of the kth row and the lth column. Moreover, the ψ refer to the eigenfunctions of (7), i.e.

$$A_{2k-1}\psi_{2k+1}^{(l)} + A_{2k-3}\psi_{2k-3}^{(l)} = \lambda_l \psi_{2k-1}^{(l)} \tag{30}$$

$$A_{2k}\psi_{2k+2}^{(l)} + A_{2k-2}\psi_{2k-2}^{(l)} = \lambda_l \psi_{2k}^{(l)} \qquad k = 1, 2, \ldots, j + \tfrac{1}{2}. \tag{31}$$

This system is the rewriting of

$$\sum_{n=0}^{j-\frac{1}{2}} H_{l,2n+1}\psi_{2n+1}^{(k)} = E_k \psi_l^{(k)} \qquad \sum_{n=0}^{j-\frac{1}{2}} H_{l,2n+2}\psi_{2n+2}^{(k)} = E_k \psi_l^{(k)} \tag{32}$$

which are the matricial forms of (19). The system (30) and (31) can be solved rather easily and we obtain

$$\psi_{2k+1}^{(l)} = \frac{F(A_{2p-1}, k-1, \lambda_l)}{A_1 A_3 \ldots A_{2k-1}} \tag{33}$$

$$\psi_{2k+2}^{(l)} = \frac{F(A_{2p}, k-1, \lambda_l)}{A_2 A_4 \ldots A_{2k}} \qquad k = 1, 2, \ldots, j - \frac{1}{2} \tag{34}$$

where the function F has been defined in (24). The remaining $\psi_1^{(l)}$ and $\psi_2^{(l)}$ have been fixed to 1. Rewriting the matrix S in (29), we then can see that its rows and columns are independent, ensuring the existence of S^{-1}.

Now that we are sure of the existence of S and its inverse, we can, as a second step, write the Hamiltonian (28) in the form

$$H' = \sum_{k=1}^{j+\frac{1}{2}} E_k (e_{k,k} + e_{j+k+\frac{1}{2}, j+k+\frac{1}{2}}) \equiv \sum_{k=1}^{j+\frac{1}{2}} E_k P'_k \tag{35}$$

where the matrices P'_k have the properties of projectors, i.e.

$$P'_k P'_l = \delta_{kl} P'_k \qquad \sum_{k=1}^{j+\frac{1}{2}} P'_k = I. \tag{36}$$

The key point is that, despite the non-unitarity of S, this property still holds if we come back to the Hamiltonian H. Indeed, we have

$$H = SH'S^{-1} = \sum_{k=1}^{j+\frac{1}{2}} E_k S P'_k S^{-1} \equiv \sum_{k=1}^{j+\frac{1}{2}} E_k P''_k \tag{37}$$

with

$$P''_k P''_l = \delta_{kl} P''_k \qquad \sum_{k=1}^{j+\frac{1}{2}} P''_k = I. \tag{38}$$

The projectors P''_k refer to the other representation of H we considered at the beginning of this section. More precisely, we can take

$$P''_k = \sigma_0 \otimes P_k \tag{39}$$

σ_0 being the 2×2 identity matrix, and it is then clear that the operators

$$Q_1 = \sum_{k=1}^{j+\frac{1}{2}} \sqrt{E_k}\, \sigma_1 \otimes P_k \tag{40}$$

$$Q_2 = \sum_{k=1}^{j+\frac{1}{2}} \sqrt{E_k}\, \sigma_2 \otimes P_k \tag{41}$$

are effective supercharges of the Hamiltonian (37) in this $N = 2$ context [8]. We have thus proved that the Hamiltonian (6) (realized through the projectors (38)) is supersymmetric when $m = n = 2$.

The last point is to realize these projectors P_k. We will achieve this by using the Clifford algebras Cl_{2M-1} [11]. Let us recall that the algebra Cl_{2M-1} is that generated by the elements α_k $(k = 1, 2, \ldots, 2M-1)$ satisfying the anticommutation relations

$$\{\alpha_k, \alpha_l\} = 2\delta_{kl}. \tag{42}$$

Then, we can convince ourselves that the projectors P_k are given by

$$P_1 = P^+ P_1^+ P_2^+ \ldots P_{M-1}^+ \tag{43}$$

$$P_2 = P^+ P_1^+ P_2^+ \ldots P_{M-1}^- \tag{44}$$

and so on, until

$$P_{j+\frac{1}{2}} = P^- P_1^- P_2^- \ldots P_{M-1}^- \tag{45}$$

where the positive integer M is fixed according to

$$j \leqslant 2^M - \tfrac{1}{2}. \tag{46}$$

In these last relations, the projectors $P^\pm$ and $P_l^\pm$ are realized through

$$P^\pm = \tfrac{1}{2}(1 \pm \alpha_1) \tag{47}$$

$$P_l^\pm = \tfrac{1}{2}(1 \pm \mathrm{i}\alpha_{2l}\alpha_{2l+1}) \qquad l = 1, 2, \ldots, M-1. \tag{48}$$

4. The ($j = \frac{5}{2}$) and ($j = \frac{7}{2}$) examples

If we take $j = \frac{5}{2}$, we have three different values for the twofold degenerated energies (23). The quantities λ_k $(k = 1, 2, 3)$ are solutions of

$$(\lambda^3 - 112\lambda)^2 = 0 \tag{49}$$

leading to

$$E_1 = 5\omega \qquad E_2 = 5\omega + 4\sqrt{7}g \qquad E_3 = 5\omega - 4\sqrt{7}g. \tag{50}$$

The Hamiltonian (6) can be realized with 6×6 matrices through the Higgs representation (7) or it can be realized with 8×8 matrices through the representation (37). In this last case, we can consider the Clifford generators of Cl_3

$$\alpha_1 = \sigma_1 \otimes \sigma_0 \qquad \alpha_2 = \sigma_2 \otimes \sigma_0 \qquad \alpha_3 = \sigma_3 \otimes \sigma_3 \tag{51}$$

giving rise to

$$P_1 = \tfrac{1}{4}(\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_0 - \sigma_1 \otimes \sigma_3 - \sigma_0 \otimes \sigma_3) \tag{52}$$

$$P_2 = \tfrac{1}{4}(\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_0 + \sigma_1 \otimes \sigma_3 + \sigma_0 \otimes \sigma_3) \tag{53}$$

$$P_3 = \tfrac{1}{2}(\sigma_0 \otimes \sigma_0 - \sigma_1 \otimes \sigma_0) \tag{54}$$

and the corresponding supercharges (40) and (41).

If we now take $j = \frac{7}{2}$, we have four different values for the energies (23). The quantities λ_k $(k = 1, 2, 3, 4)$ become solutions of

$$(\lambda^4 - 504\lambda^2 + 15120)^2 = 0 \tag{55}$$

leading to

$$\begin{aligned} E_1 &= 7\omega + \sqrt{252 + 48\sqrt{21}}g \qquad & E_2 &= 7\omega + \sqrt{252 - 48\sqrt{21}}g \\ E_3 &= 7\omega - \sqrt{252 + 48\sqrt{21}}g \qquad & E_4 &= 7\omega - \sqrt{252 - 48\sqrt{21}}g. \end{aligned} \tag{56}$$

The Hamiltonian (6) can be realized with 8×8 matrices both in the Higgs representation (7) and in the representation (37). The realization (51) still holds, leading to

$$P_3 = \tfrac{1}{4}(\sigma_0 \otimes \sigma_0 - \sigma_1 \otimes \sigma_0 - \sigma_1 \otimes \sigma_3 - \sigma_0 \otimes \sigma_3) \tag{57}$$

$$P_4 = \tfrac{1}{4}(\sigma_0 \otimes \sigma_0 - \sigma_1 \otimes \sigma_0 + \sigma_1 \otimes \sigma_3 + \sigma_0 \otimes \sigma_3) \tag{58}$$

besides the projectors (52) and (53). The corresponding supercharges (40) and (41) are then once again completely determined.

5. Comments

We have proved, by using a deformed $sl(2, R)$ algebra and specific projectors, that the Hamiltonian (6) is supersymmetric when $m = n = 2$. Let us immediately stress that this does not mean that this Hamiltonian is not supersymmetric for other values of m and n. Simply in these cases the representations of the deformed algebra (10) and (11) are not yet known and they have first to be performed before going to physical conclusions such as supersymmetry.

Another comment is that it is the first time, to our knowledge, that a Hamiltonian of quantum optics has been found to be supersymmetric in itself. We have to distinguish our approach, where only *one* Hamiltonian is concerned, from other approaches [12] where *two* Hamiltonians (with identical spectra) are necessary in order to reveal supersymmetric features in quantum optics.

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